A note on configurations of lines

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§ 1. INTRODUCTION

The main goal of this note is a correction to a result of Gallagher and Proulx ([1], lemma 3 and the following corollary). To a system of 1-dimensional subspaces $W_1, \ldots, W_k$ of an n-dimensional real or complex Hilbert space V they associate a graph with vertices $1, \ldots, k$ and edges consisting of ordered pairs (i, j) with $i \neq j$ for which W_i and W_j are not orthogonal. We show that the maximal length of minimal circuits in such a graph is $n+2$ $(n \geqslant 2)$.

Given two systems $W_1, \ldots, W_k$ and $W_1', \ldots, W_k'$ of 1-dimensional subspaces of an n-dimensional Hilbert space V, Gallagher and Proulx discuss necessary and sufficient conditions for the existence of a linear isometry of V sending W_i to W_i' for each i. They prove that not only the angle θ_{ij} between W_i and W_j must be the same as the angle between W_i' and W_j', but that also certain cohomology classes must be the same ([1], theorem 2). The decisive cohomology class is a homomorphism φ from the multiplicative group generated by the circuits (the homology group) of the graph to the group U of unimodular scalars.

Let $w_1, \ldots, w_k$ be unit vectors, with $w_i \in W_i$ for each i. For each edge (i, j) of the graph the inner products are $\langle w_i, w_j \rangle = z_{ij} \cdot \cos \theta_{ij}$ with $z_{ij} \in U$ and $\cos \theta_{ij} > 0$. The value of φ on a minimal circuit of the form $\gamma = (j_1 j_2)(j_2 j_3) \ldots (j_m j_1)$ is now defined by $\varphi(\gamma) = z_{j_1 j_2} \ldots z_{j_m j_1}$; this is independent of the choice of the unit vectors.

In addition to the results of Gallagher and Proulx we shall prove here that for the *real* case the equality of the above cohomology classes need only be checked on minimal circuits of length $\leqslant n$, and for the *complex* case only on minimal circuits of length $\leqslant n+1$.

I want to thank J. J. Seidel for drawing my attention to this subject, and for his careful reading of a preliminary version.

§ 2. ORTHOGONS

In the sequel I shall work in n-space $\mathbb{F}^n$, where $\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$, with standard inner product $\langle\, ,\rangle$. In this setting the correct version of [1], lemma 3 should read as follows:

2.1 THEOREM. The homology group of the graph associated to a system of 1-dimensional subspaces $W_1, \ldots, W_k$ in $\mathbb{F}^n$ of dimension $n \geqslant 2$ is generated by the minimal circuits $\gamma = (j_1 j_2) \ldots (j_m j_1)$ with $m \leqslant n+2$.

In the proof of this theorem I found it convenient to use the following concept:

2.2 DEFINITION. An ordered set of vectors $v_1, \ldots, v_m$ in $\mathbb{F}^n$ will be called an *m-orthogon* whenever the inner products satisfy $\langle v_i, v_j\rangle \neq 0$ iff $i-j \equiv -1, 0, 1 \pmod{m}$.

Note that if $v_1, \ldots, v_m$ is an orthogon then $v_i \neq 0$. In general v_i need not be different from v_j; however, if $m \geqslant 4$ then for different i, j the pair v_i, v_j is linearly independent. Choosing a non-null vector in each 1-dimensional subspace, the relation between m-orthogons and minimal circuits of length m in the chain group of the associated graph is obvious: For every minimal circuit of length m there exists an m-orthogon, and every pairwise linearly independent m-orthogon gives rise to a set of lines such that its graph is a minimal circuit. The key result on the existence of m-orthogons in $\mathbb{F}^n$ is the following lemma.

2.3 LEMMA. Suppose $n \geqslant 1$. Then $\mathbb{F}^n$ contains an m-orthogon if and only if $\mathbb{F}^{n+1}$ contains an $(m+1)$-orthogon.

Since $\mathbb{F}^1$ contains no two non-null mutually orthogonal vectors, for an m-orthogon in $\mathbb{F}^1$ we have maximally $m=3$, which is realised by taking for example $v_1 = v_2 = v_3$. Inductive application of lemma 2.3 yields that for an m-orthogon in $\mathbb{F}^n$ we always have $m \leqslant n+2$, while $m = n+2$ can be realised. Taking into account these remarks and the above note on the correspondence between orthogons and minimal circuits, the proof of lemma 2.3 will immediately prove theorem 2.1.

PROOF OF LEMMA 2.3. Since $\mathbb{F}^n$ always contains a 3-orthogon, we may suppose that $m \geqslant 3$. Now let $v_1, v_2, v_3, \ldots, v_m$ be an m-orthogon in $\mathbb{F}^n$. Embed $\mathbb{F}^n$ in a standard way in $\mathbb{F}^{n+1}$, and choose a unit vector w_0 in

$\mathbf{F}^{n+1}$, orthogonal to the subspace $\mathbf{F}^n$. Then consider in $\mathbf{F}^{n+1}$ the ordered set

$$v_1 - \langle v_1, v_2\rangle w_0,\ w_0,\ v_2 + w_0,\ v_3,\ \ldots,\ v_m.$$

Taking inner products this set is shown to be an $(m+1)$-orthogon.

On the other hand, let $v_1, v_2, v_3, v_4, \ldots, v_k$ be a k-orthogon in $\mathbf{F}^{n+1}$. Without loss of generality we may suppose that v_2 is a unit vector. Now identify $\mathbf{F}^n$ with the orthocomplement of v_2 in $\mathbf{F}^{n+1}$ and consider the ordered set

$$v_1 - \langle v_1, v_2\rangle v_2,\ v_3 - \langle v_3, v_2\rangle v_2,\ v_4,\ \ldots,\ v_k.$$

Since $\langle v_1, v_3\rangle = 0$, this set clearly is a $(k-1)$-orthogon, and the whole set lies in $\mathbf{F}^n$. *End of the proof.*

The actual proof of theorem 2.1 is now left to the reader.

§ 3. ORTHOGONS, ORTHOCHAINS AND GRAMIANS

3.1 DEFINITION. An ordered set of vectors $v_1, \ldots, v_m$ in $\mathbf{F}^n$ will be called a *simple m-orthochain* whenever $\langle v_i, v_j\rangle \neq 0$ iff $i-j=-1$, 0 or 1.

REMARK. Deleting one vector from an m-orthogon leaves a simple $(m-1)$-orthochain (possibly the ordering of the remaining set of vectors has to be rearranged).

CONVENTION. In this section we shall always assume that orthogons and simple orthochains are composed of *vectors* of *unit length*. This is no loss of generality, but it simplifies the computations considerably.

NOTATION. The determinant of the Gramian matrix of a simple orthochain $v_1, \ldots, v_m$ will be denoted by $D(v_1, \ldots, v_m)$, while the determinant of the Gramian matrix of an orthogon $v_1, \ldots, v_m$ will be denoted by $C(v_1, \ldots, v_m)$.

Expansion by minors of the last column yields the following formulas for $m > 3$:

3.2 $$D(v_1, \ldots, v_m) = D(v_1, \ldots, v_{m-1}) - |\langle v_{m-1}, v_m\rangle|^2 D(v_1, \ldots, v_{m-2})$$

3.3 $$\begin{aligned} C(v_1, \ldots, v_m) &= D(v_1, \ldots, v_{m-1}) - |\langle v_1, v_m\rangle|^2 D(v_2, \ldots, v_{m-1}) - \\ &- |\langle v_{m-1}, v_m\rangle|^2 D(v_1, \ldots, v_{m-2}) + \\ &+ 2\cdot(-1)^{m-1}\cdot(\operatorname{Re}\varphi(\gamma))\cdot|\langle v_1, v_2\rangle| \ldots |\langle v_{m-1}, v_m\rangle|\cdot|\langle v_m, v_1\rangle|, \end{aligned}$$

where γ denotes the minimal circuit $(1, 2)(2, 3) \ldots (m-1, m)\ (m, 1)$ associated to $v_1, \ldots, v_m$ and where $\operatorname{Re}\varphi(\gamma)$ denotes the real part of $\varphi(\gamma)$.

From these formulas one can deduce many results, among which I mention the following lemmas:

3.4 LEMMA. In a simple orthochain $v_1, \ldots, v_m$ the vectors $v_1, \ldots, v_{m-1}$ form a linearly independent set.

PROOF. Suppose $x=\sum_{i=1}^{k}\alpha_i v_i=0$ $(k<m-1,\ \alpha_k\neq 0)$. Then $\langle x, v_{k+1}\rangle =$ $=\alpha_k\langle v_k, v_{k+1}\rangle=0$. Thus $\alpha_k=0$, contradiction.

3.5 LEMMA. For an m-orthogon in $\mathbf{F}^n$ with $m>n$, the number $\operatorname{Re}\varphi(\gamma)$ is determined by the angles between the lines spanned by the vectors.

PROOF. In formula 3.3 we have $C(v_1, \dots, v_m)=0$ since $m>n$, whereas from 3.2 it follows by induction that all D-terms in 3.3 depend only on the angles between the lines.

REMARK. Lemma 3.5 cannot be strengthened. To illustrate this we consider the following two 3-orthogons in $\mathbf{F}^3$:

$$v_1=\tfrac{1}{2}\sqrt{2}(1,1,0),\ v_2=\tfrac{1}{2}\sqrt{2}(1,0,1),\ v_3=\tfrac{1}{2}\sqrt{2}(0,1,1)$$
$$v_1'=v_1,\ v_2'=v_2,\ v_3'=\tfrac{1}{2}\sqrt{2}(0,-1,1).$$

Then $|\langle v_i, v_j\rangle|=|\langle v_i', v_j'\rangle|$, but $\varphi(\gamma)=1$ and $\varphi(\gamma')=-1$.

3.6 COROLLARY. For an $(n+2)$-orthogon $v_1, \dots, v_{n+2}$ in $\mathbf{C}^n$ the number $\varphi(\gamma)$ is a real number.

PROOF. $\varphi(\gamma)$ is independent of scalar multiplication of the vectors with unimodular complex numbers. Therefore we may assume that all inner products $\langle v_i, v_j\rangle$ are *real*, except possibly $\langle v_{n+1}, v_{n+2}\rangle$. Now $v_{n+2}=\xi_1 v_1+$ $+\dots+\xi_n v_n$ since $v_1, \dots, v_{n+1}$ is a simple orthochain and so by lemma 3.4 $v_1, \dots, v_n$ is a basis for $\mathbf{C}^n$. Taking now successively inner products $\langle v_{n+2}, v_i\rangle$, $i=n+2, 1, 2, \dots, n-1$, it appears that all ξ_i are real numbers. Therefore $\langle v_{n+1}, v_{n+2}\rangle$ is a real number, which proves the corollary.

REMARK. There is no analogous result to 3.6 in the case of an $(n+1)$-orthogon $v_1, \dots, v_{n+1}$ in $\mathbf{C}^n$, as can be shown by the following example in $\mathbf{C}^2$:

If $v_1=(1,0)$, $v_2=1/\sqrt{2}(1,1)$ and $v_3=1/\sqrt{2}(1,i)$ then $\varphi(\gamma)=1/\sqrt{2}(1-i)$. This example also shows that in lemma 3.5 for $m=n+1$ it is essential to take the real part of $\varphi(\gamma)$. Indeed, from the given orthogon we obtain another by complex conjugation of the vectors. This does not affect the angles between the lines, but the cohomology class must be replaced by its complex conjugate.

3.7 THEOREM.

(i) For an $(n+2)$-orthogon in $\mathbf{F}^n$ the value of $\varphi(\gamma)$ equals $(-1)^{n+1}$.

(ii) For an $(n+1)$-orthogon in $\mathbf{R}^n$ the value of $\varphi(\gamma)$ depends only on the angles between the lines spanned by the vectors.

PROOF. (i) Let $v_1, \dots, v_{n+2}$ be an orthogon in $\mathbf{F}^n$. Since

$$C(v_1, \dots, v_{n+2})=D(v_1, \dots, v_{n+1})=0,$$

formula 3.3 ($m=n+2$) shows that $(-1)^{n+1} \cdot \operatorname{Re}\varphi(\gamma)>0$. Moreover, $\varphi(\gamma)$ is a unimodular real number by corollary 3.6.

(ii) See lemma 3.5.

Because of the above results, the corollary on page 161 of [1] can be corrected as follows.

3.8 THEOREM. A system of 1-dimensional subspaces $W_1, \ldots, W_k$ in $\mathbf{F}^n$ of dimension $n>2$ is determined up to a linear isometry of $\mathbf{F}^n$ by the numbers

$$tr(P_{j_1} \ldots P_{j_m}) \ (1<j_u<k, \ m<n+1),$$

where P_j is the orthogonal projection of $\mathbf{F}^n$ onto W_j. In case of real space $\mathbf{R}^n$ it suffices to consider $m<n$.

REFERENCE

1. Gallagher, P. X. and R. J. Proulx – Orthogonal and unitary invariants of families of subspaces. In BASS et al., Contributions to Algebra: A collection of papers dedicated to Ellis Kolchin. Academic Press, New York etc. 1977. pp. 157–164.